Lecture 8
Transverse multibunch instabilities in circular accelerators

January 24, 2019

Lecture outline

- Transverse multi-bunch instabilities
- Resistive wall transverse coupled-bunch instability

Long-range and short-range wake fields

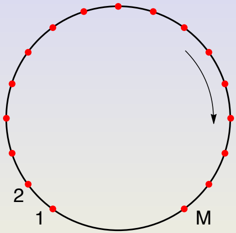
Depending on the source of the wake field, the wake function can fall off rapidly with distance, on a scale comparable to the length of a single bunch. Such "short-range" wake fields are important for single-bunch instabilities like BBU studied in the previous lecture.
In other cases, the wake function extends over the distance from one bunch to another. This is a "long-range" wake field; it can drive coupled-bunch instabilities which are the subject of this lecture.

In case of long-range wake fields, we can often treat each bunch as a single "macroparticle". This requires:

- the distance between the bunches to be much larger than the length of the bunches;
- all particles in a given bunch see the same wake field, and respond to it in the same way;
- the bunch centroid can move, but the distribution of the bunch around the centroid remains unchanged.

Equation of motion for betatron oscillations

We now study a transverse coupled bunch instability of a train of bunches in a circular accelerator.



M bunches in a ring.

In the absence of any wake fields, the equation of motion for the nth bunch moving in a storage ring can be written as

$$\ddot{y}_n + \omega_\beta^2 y_n = 0 \tag{8.1}$$

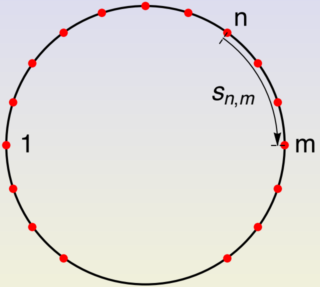
where y_n is the transverse offset of bunch n and $\ddot{y} = d^2y/dt^2$. The "averaged" betatron frequency is

$$\omega_\beta = \frac{2\pi\nu_\beta}{T} \tag{8.2}$$

where ν_β is the betatron tune and $T = C/c$ is the revolution period, C is the circumference of the ring (we use a smooth focusing approximation for the betatron oscillations).

Account of transverse wake

We now add transverse forces from the wake fields as driving terms on the right-hand side of the equation of motion. Bunches are considered as point charges.



Consider bunch n and another bunch, m, moving ahead of n at a distance $s_{n,m}$. Bunch m generates the transverse wake *per unit length* $\bar{w}_t(s_{n,m})y_m$. With account of the wake field transverse force $(= Qe\bar{w}_t(s_{n,m})y_m)$

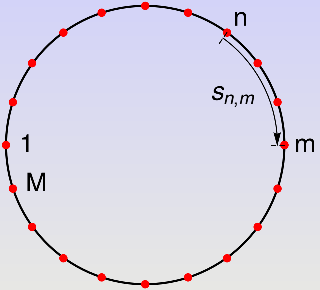
$$\ddot{y}_n(t) + \omega_\beta^2 y_n(t) = \frac{Qe}{m\gamma}\bar{w}_t(s_{n,m})y_m$$

We need to take y_m at time when the driving particle was at the current location of y_n, that is $y_m(t - s_{n,m}/c)$[24],

$$\ddot{y}_n(t) + \omega_\beta^2 y_n(t) = \frac{Qe}{m\gamma}\bar{w}_t(s_{n,m})y_m\left(t - \frac{s_{n,m}}{c}\right) \tag{8.3}$$

[24] Think about wake in a cavity.

Many bunches, account of earlier revolutions



Consider now M bunches *uniformly distributed* in a ring. The bunches are counted from 1 to M, where the first bunch is at the tail of the train, and the M-th bunch is located at the head (the bunch count increases in the direction of motion)

$$\ddot{y}_n(t) + \omega_\beta^2 y_n(t) = A \sum_{k=0}^{\infty} \sum_{m=1}^{M} \bar{w}_t(s_{n,m} + Ck) y_m \left(t - \frac{s_{n,m}}{c} - Tk \right) \qquad (8.4)$$

Here $s_{n,m}$ is the distance between the bunch n and bunch m, which is measured along the circle in the direction of motion from bunch n to m; $s_{n,m}$ is a positive number. The distance $s_{n,n+1}$ is equal to the interbunch distance s_b, the distance $s_{n,n-1}$ is equal to $(M-1)s_b$. The wake is defined so that $\bar{w}(s) = 0$ for $s < 0$. The value of $k = 0$ corresponds to the wake that is generated during the current revolution; $k > 0$ give the wake that was created on previous revolutions.

Change summation limits

The parameter A is

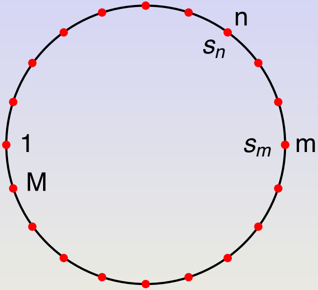
$$A = \frac{Ne^2}{m\gamma}$$

where N is the number of particles in each bunch.
Eq. (8.4) is written so that the argument in the wake function is always positive. We will now modify it using the property of the wake $\bar{w}_t(s) = 0$ for $s < 0$. This allows us to extend the summation over k from $-\infty$ to ∞,

$$\sum_{k=0}^{\infty} \to \sum_{k=-\infty}^{\infty},$$

because the added new terms have a negative argument in the wake and hence do not contribute to the sum.

Manipulate equations



We can now redefine $s_{n,m}$ considering it as a difference of longitudinal positions s_m and s_n of the two bunches in the ring, $s_{n,m} = s_m - s_n$ (s_m and s_n are measured along the beam path from an arbitrary established origin $s = 0$). If $m > n$, then $s_m > s_n$ and the formula gives the correct value for $s_{n,m}$. For $m < n$, we have $s_m < s_n$ and $s_{n,m} = C + s_m - s_n$.

However, since we carry summation over k from $-\infty$ to ∞, the additional term C in $s_{n,m}$ does not change the result, because it can be eliminated by the change of the summation variable $k \to k+1$. Hence,

$$\ddot{y}_n(t) + \omega_\beta^2 y_n(t) = A \sum_{k=-\infty}^{\infty} \sum_{m=1}^{M} \bar{w}_t(s_m - s_n + Ck) y_m \left(t - \frac{s_m - s_n}{c} - Tk \right) \tag{8.5}$$

The frequency ω of the oscillations

Since we have a linear system of equations, we seed a solution $y_n = \hat{y}_n e^{-i\omega t}$. A mode with $Im\omega > 0$ means an instability. This is the *coupled-bunch transverse instability* (CBTI). The frequency ω satisfies the dispersion relation

$$(\omega_\beta^2 - \omega^2)\hat{y}_n = A \sum_{k,m} \bar{w}_t(s_m - s_n + Ck)\hat{y}_m e^{i\omega(\tau_m - \tau_n) + i\omega Tk} \tag{8.6}$$

Here we introduced a new variable $\tau_n = s_n/c$.

Carry our summation

We can carry out the summation over k in Eq. (8.6) by using a periodic δ-function

$$\sum_{k=-\infty}^{\infty} \delta(x-k) = \sum_{j=-\infty}^{\infty} e^{2\pi ijx} \tag{8.7}$$

We have

$$\begin{aligned}
I &\equiv \sum_k \bar{w}_t(s_m - s_n + Ck)e^{i\omega Tk} \\
&= \int_{-\infty}^{\infty} dx\, \bar{w}_t(s_m - s_n + Cx)e^{i\omega Tx} \sum_{k=-\infty}^{\infty} \delta(x-k) \\
&= \int_{-\infty}^{\infty} dx\, \bar{w}_t(s_m - s_n + Cx)e^{i\omega Tx} \sum_{j=-\infty}^{\infty} e^{2\pi ijx} \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, \bar{w}_t(s_m - s_n + Cx)e^{i\omega Tx + 2\pi ikx}.
\end{aligned}$$

The last integral can be expressed in terms of the transverse impedance $Z_t(\omega)$.

Express through impedance

See the definition (4.5)

$$Z_t(\omega) = -\frac{i}{c}\int_{-\infty}^{\infty} dz\,\bar{w}_t(z)e^{i\omega z/c}$$

This yields $\xi = Cx + s_m - s_n$,

$$\begin{aligned} I &= \frac{1}{C}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi\,\bar{w}_t(\xi)\exp\left[i\frac{\xi-(s_m-s_n)}{c}(\omega+k\omega_0)\right] \\ &= \frac{ic}{C}\sum_{k=-\infty}^{\infty}\exp\left[-i(\tau_m-\tau_n)(\omega+k\omega_0)\right]Z_t(\omega+k\omega_0) \end{aligned}$$

The term with the frequency ω cancels a term in Eq. (8.6). Replace $(\tau_m - \tau_n)\omega_0 = 2\pi(m-n)/M$,

$$(\omega_\beta^2 - \omega^2)\hat{y}_n = \frac{icA}{C}\sum_{k,m} e^{-2\pi ik(m-n)/M} Z_t(\omega + k\omega_0)\hat{y}_m$$

The final answer

There are different modes, or patterns, of oscillations. Each mode is marked by an integer value of p, $p = 0, 1, 2, \ldots, M-1$. The solution corresponding to one mode is sought as $\hat{y}_n = Y_p e^{2\pi i p n/M}$. The frequency of the p-th mode ω_p is found from

$$\omega_\beta^2 - \omega_p^2 = \frac{icA}{C} \sum_{k=-\infty}^{\infty} Z(\omega_p + k\omega_0) \sum_{m=1}^{M} e^{-2\pi i (k-p)(m-n)/M}$$

The last sum is equal to zero unless $k = Mq + p$, where q is an integer, in which case the sum is equal to M. This reduces the dispersion relation to the following form

$$\omega_\beta^2 - \omega_p^2 = \frac{iAM}{T} \sum_{q=-\infty}^{\infty} Z_t(\omega_p + p\omega_0 + Mq\omega_0)$$

To find the frequency shift $\Delta\omega_p = \omega_p - \omega_\beta$, assume $|\Delta\omega_p| \ll \omega_\beta$ and use $\omega_\beta^2 - \omega_p^2 \approx -2\Delta\omega_p\omega_\beta$.

More convenient form for the frequency shift

$$\Delta\omega_p = -\frac{iM}{2T\omega_\beta}\frac{Ne^2}{m\gamma}\sum_{q=-\infty}^{\infty} Z_t(\omega_\beta + p\omega_0 + Mq\omega_0) \qquad (8.8)$$

Use the beam current in the ring I and the Alvfen current I_A

$$I = \frac{MNe}{T}, \qquad I_A = \frac{4\pi}{Z_0 c}\frac{mc^3}{e} = 17.045\ \mathrm{kA} \qquad (8.9)$$

We can go beyond the smooth focusing approximation replacing $\omega_\beta = 2\pi\nu_\beta/T$ with

$$2\pi\nu_\beta = \int \frac{ds}{\beta_\perp} \to \frac{C}{\langle\beta_\perp\rangle} \qquad (8.10)$$

We then obtain

$$\Delta\omega_p = -i\frac{4\pi}{Z_0}\frac{c}{2\gamma}\frac{I}{I_A}\langle\beta_\perp\rangle\sum_q Z_t(\omega_\beta + p\omega_0 + Mq\omega_0)$$

Reminder: Z_t is the impedance per unit length.

Impedance varies around the ring

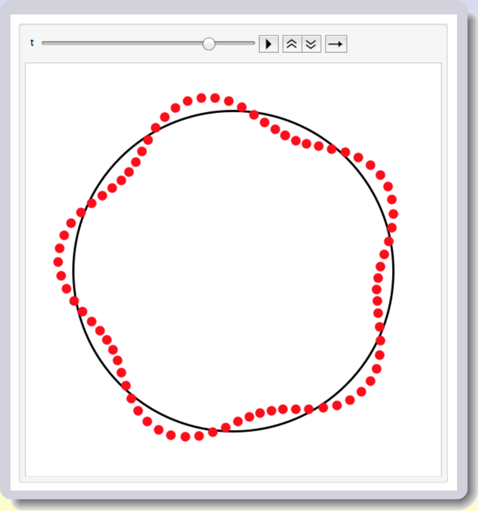
In practical situations the impedance varies from one location in the ring to another. As more rigorous analysis gives in this case

$$\Delta\omega_p = -i\frac{4\pi}{Z_0}\frac{c}{2\gamma}\frac{I}{I_A}\sum_q \langle \beta_\perp Z_t(\omega_p + p\omega_0 + Mq\omega_0)\rangle$$

Impedance at locations with large value of the beta-function is more harmful for stability.

Animation

See the animation. Note the direction of propagation of the wave for $p = p_0$ and $p = M - p_0$.



CBTI for resistive wall wake

We now derive CBTI for the coupled bunch instability driven by the resistive wall wake. We start from Eq. (8.8):

$$\Delta\omega_p = -\frac{iM}{2CT\omega_\beta}\frac{Ne^2}{m\gamma}\sum_{q=-\infty}^{\infty} Z_t^\circ[\omega_\beta + (qM+p)\omega_0]\,, \tag{8.11}$$

where $Z_t^\circ = CZ_t$ is the transverse impedance for the whole ring.
We can carry out the summation analytically, if we use wake fields instead of impedances. In terms of wake fields, Eq. (8.11) can be written as follows

$$\Delta\omega_p = -\frac{Ne^2}{2mc\gamma T\omega_\beta}\sum_{n=1}^{\infty} \bar{w}_t^\circ(ns_b)e^{2\pi i(p+\nu_\beta)n/M} \tag{8.12}$$

where s_b is the distance between the bunches, $s_b = C/M$ and $\bar{w}_t^\circ$ the wake for the ring $(= C\bar{w}_t)$.

CBTI for resistive wall wake

Here is the proof of this. We can extend the summation over n to $-\infty$, because the wake is zero there:

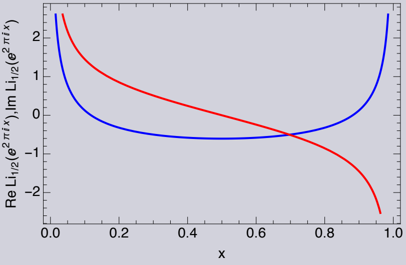
$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \bar{w}_t^{\circ}(ns_b)e^{2\pi i(p+\nu_\beta)n/M} \\
&= \int_{-\infty}^{\infty} ds\, \bar{w}_t^{\circ}(s)e^{2\pi i(p+\nu_\beta)s/s_bM} \sum_{n=-\infty}^{\infty} \delta(s-ns_b) \\
&= \int_{-\infty}^{\infty} ds\, \bar{w}_t^{\circ}(s)e^{2\pi i(p+\nu_\beta)s/s_bM} \frac{1}{s_b} \sum_{q=-\infty}^{\infty} e^{2\pi iqs/s_b} \\
&= \sum_{q=-\infty}^{\infty} \frac{ic}{s_b} Z_t^{\circ}(\omega_0(p+\nu_\beta)+\omega_0 Mq)
\end{aligned} \tag{8.13}$$

CBTI for resistive wall wake

For the resistive wall the transverse wake decays with distance as $\bar{w}_t^\circ(s) = Ds^{-1/2}$ (see Eq. (5.16)) where D is a constant, and the sum can be computed analytically in terms of the polylogarithm function $\mathrm{Li}_{\frac{1}{2}}(x)$: $\mathrm{Li}_{\frac{1}{2}}(x) = \sum_{n=1}^{\infty}(x^n/\sqrt{n})$, so that

$$\Delta\omega_p = -\frac{Ne^2}{2mc\gamma T\omega_\beta}\bar{w}_t^\circ(s_b)\mathrm{Li}_{\frac{1}{2}}\left(e^{2\pi i(p+\nu_\beta)/M}\right) \tag{8.14}$$

The function $\mathrm{Im}\,\mathrm{Li}_{\frac{1}{2}}(e^{2\pi i x})$ is a periodic function with the period equal to 1. It has a singularity when $x \to 0$. Its imaginary part diverges, $\mathrm{Im}\,\mathrm{Li}_{\frac{1}{2}}(x) \to +\infty$, when $x \to +0$ and $\mathrm{Im}\,\mathrm{Li}_{\frac{1}{2}}(x) \to -\infty$, when $x \to -0$. Blue line—real, red line—imaginary parts of $\mathrm{Li}_{\frac{1}{2}}(e^{2\pi i x})$.

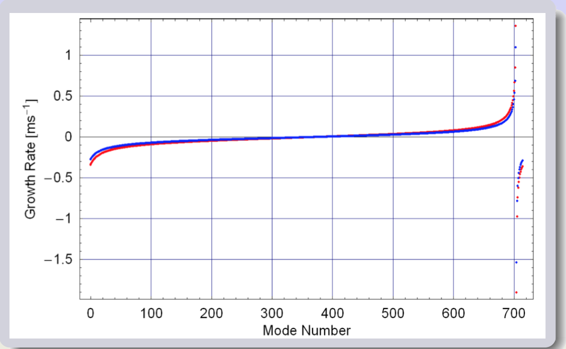


CBTI for resistive wall wake

This means that the maximum growth rate is attained for the minimal value of the argument $(p+\nu_\beta)/M$, when it is negative. The most unstable modes have negative $p=-(1+\{\nu_\beta\})$ where $\{\nu_\beta\}$ is the integer part of the tune with the argument of the $\mathrm{Li}_{\frac{1}{2}}$ function equal to $-(1-[\nu_\beta])/M$, where $[\nu_\beta]$ is the fractional part of the tune. For small negative values of the argument x, the function $\mathrm{Li}_{\frac{1}{2}}(e^{2\pi i x})$ can be approximated by $(1-i)/2\sqrt{-x}$ which gives the following equation for the approximate value of the growth rate of the instability

$$\mathrm{Im}\,\Delta\omega_p = \frac{Ne^2}{4mc\gamma T\omega_\beta}\bar{w}_t^\circ(s_b)\sqrt{\frac{M}{1-[\nu_\beta]}} \tag{8.15}$$

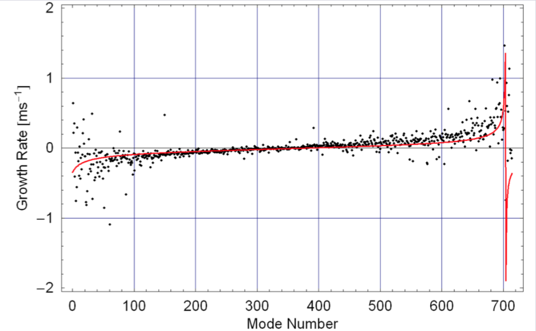
CBTI for resistive wall wake



Growth rates of vertical coupled-bunch modes in the NLC Main Damping Rings[25]. The ring is assumed to be uniformly filled with 714 bunches. Red points show the growth rates assuming the nominal bunch charge. Blue points are correspond to a uniform fill, and red point to a non-uniform one.

[25] A. Wolski. Paper "Resistive Wall Instability in the NLC Main Damping Rings", http://repositories.cdlib.org/lbnl/LBNL-59526

CBTI for resistive wall wake



Growth rates of various modes in a uniformly filled ring. Black points: tracking simulation. Red line: analytical estimate. Modes with positive growth rates are unstable.